HOFER METRIC FROM THE CONTACT POINT OF VIEW

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ABSTRACT. Given a manifold M endowed with a contact 1-form α, a bi-invariant pseudo-metric ϱ_α is introduced on $\mathrm{Cont}_0(M,\alpha)$, the compactly supported identity component of the group of all strict contactomorphisms of (M,α). For M open ϱ_α is a metric. If (N,ω) is a symplectic manifold which is either closed integral or open exact, and if (M,α) is the prequantization space of (N,ω), the existence of ψ in the commutator subgroup of $\mathrm{Cont}_0(M,\alpha)$ such that $\varrho_\alpha(\mathrm{id},\psi) > 0$ is shown by elementary methods. In view of the simplicity theorem on the Hamiltonian symplectomorphism group it follows an alternative, elementary and natural, proof of the non-degeneracy of the Hofer metric ϱ_H on the Hamiltonian group $\mathrm{Ham}(N,\omega)$ without using any hard symplectic methods. Next the unboundedness of ϱ_H on $\mathrm{Ham}(N,\omega)$ is established. Finally, some estimates on the Hofer metric, which cannot be derived from the energy-capacity inequality, are given.

1. INTRODUCTION

Let $(M, \xi = \ker(\alpha))$ be a co-oriented contact manifold, i.e. M is a C^∞ smooth paracompact manifold of dimension $2n + 1$, and α is a C^∞ 1-form on M such that $\nu_\alpha = \alpha \wedge (d\alpha)^n$ is a volume form. A contactomorphism f of (M,α) is a C^∞ diffeomorphism of M such that $f^*\alpha = \lambda_f \alpha$, where λ_f is a smooth nowhere vanishing function on M depending on f and α. In other words, f preserves the contact distribution ξ. Next, a contactomorphism f is called *strict* if λ_f is equal to 1 on M.

Let $\mathrm{Cont}(M, \xi)$ (resp. $\mathrm{Cont}(M,\alpha)$) denote the group of all contactomorphisms (resp. strict contactomorphisms) of (M,ξ) (resp. (M,α)), and let $\mathrm{Cont}_0(M,\xi)$ be the compactly supported identity component of $\mathrm{Cont}(M,\xi)$. In view of [17] $\mathrm{Cont}(M,\xi)$ is a simple group. We shall deal mainly with contact manifolds (M,α) in the narrow sense. The symbol $\mathrm{Cont}_0(M,\alpha)$ stands for the totality of all elements of $\mathrm{Cont}(M,\alpha)$ which can be joined to the identity by a

compactly supported smooth isotopy in $\mathrm{Cont}(M, \alpha)$. Note that $\mathrm{Cont}_0(M, \alpha)$ is neither a simple group nor a normal subgroup of $\mathrm{Cont}_0(M, \xi)$.

The Hofer metric ϱ_H is a bi-invariant metric on the group of compactly supported Hamiltonian symplectomorphisms of a symplectic manifold. Hofer geometry constitutes a basic tool in symplectic topology (see [9], [12], [13], [16]). In attempt to extend methods of Hofer geometry to the contact case Banyaga and Donato in [2] introduced a bi-invariant metric ϱ_{BD} on $\mathrm{Cont}(M, \alpha)$ for a very special type of (M, α). Recently, Müller and Spaeth in [14] generalized this definition for all contact manifolds (M, α). In this paper we deal with a bi-invariant pseudo-metric ϱ_α on $\mathrm{Cont}_0(M, \alpha)$. For M open ϱ_α is a metric. It is then equivalent to the Banyaga-Donato metric but very likely different than it. For M open the non-degeneracy of ϱ_α is shown by making use of the energy-capacity inequality for contact manifolds [14].

For any $F \in \mathrm{C}_c^\infty(M, \mathbb{R})$ denote

$$\| F \|_\infty = \max_{p \in M} F(p) - \min_{p \in M} F(p).$$

Theorem 1.1. *Given any contact manifold (M, α) and a compactly supported vector field on M, denote*

$$\| X \|_\alpha = \| \alpha(X) \|_\infty .$$

Then the function $\varrho_\alpha : \mathrm{Cont}_0(M, \alpha) \times \mathrm{Cont}_0(M, \alpha) \to [0, \infty)$, given by

$$\varrho_\alpha(\varphi, \psi) = \inf\{l_\alpha(\{f_t\})| \{f_t\} \text{ lies in } \mathrm{Cont}_0(M, \alpha) \text{ with } f_0 = \varphi, f_1 = \psi\}$$

for any $\varphi, \psi \in \mathrm{Cont}_0(M, \alpha)$, is a bi-invariant pseudo-metric on the group $\mathrm{Cont}_0(M, \alpha)$. Here $l_\alpha(\{f_t\})$ is the length of an isotopy $\{f_t\}$ in $\mathrm{Cont}_0(M, \alpha)$ with respect to the norm $\| \cdot \|_\alpha$. Moreover, ϱ_α is a metric if and only if M is open.

Our next aim is to show some part of Theorem 1.1 without appealing to hard symplectic methods. This part is formulated as the following

Claim 1.2. *Assume that either (1) (M, α) is the total space of a prequantization bundle of a integral closed symplectic manifold (N, ω), or (2) (M, α) is the contactisation of an exact open symplectic manifold (N, ω). Then there is $\varphi \in \mathrm{Cont}_0(M, \alpha)$ such that $\varrho_\alpha(\mathrm{id}, \varphi) > 0$. Moreover, φ can be chosen in the commutator subgroup of $\mathrm{Cont}_0(M, \alpha)$.*

A straightforward consequence of Claim 1.2 and of the classical simplicity theorem on the Hamiltonian symplectomorphism group due to Banyaga [1] (Theorem 6.2) is an elementary and natural proof of the non-degeneracy of the Hofer metric ϱ_H for $\mathrm{Ham}(N, \omega)$, whenever (N, ω) is either integral closed or exact open. This is a striking enough phenomenon, bearing in mind hard symplectic methods in the classical proofs, see [9], [15], [11], [13] and [16]. This

proof constitutes an example of the influence of the structure of strict contactomorphism group (i.e. the quantomorphism group) on symplectic topology. Up to now only the opposite influence has been known, e.g. in the paper [5] by Eliashberg and Polterovich, or in [14], the contactomorphism group has been investigated by symplectic methods. Further investigations of the properties of ϱ_α might be of interest both in contact and symplectic topology. Namely, the fact that the non-degeneracy of the Hofer metric on $\mathrm{Ham}(N, \omega)$ is encoded in the related contact structure seems to reveal new perspectives in contact topology and new interrelations between both topologies.

As a by-product of the proof of Claim 1.2 we obtain some estimate on $\varrho_H(\mathrm{id}, \varphi)$, where φ is a Hamiltonian symplectomorphism determined by a "typical" bump function on $\mathbb{R}^{2n}$. It follows that the metrics ϱ_α and ϱ_H on the groups under consideration are unbounded.

Namely, for every $a, b, c > 0$ choose a smooth bump function

$$\mu = \mu_{a,b,c} : (-(a+b), a+b) \to [0, c]$$

such that $\mathrm{supp}(\mu) \subset (-(a + \frac{1}{2}b), a + \frac{1}{2}b)$, $\mu = c$ on $[-a, a]$, with "typical" properties (see section 7.2). For all $p = (x, y) \in \mathbb{R}^{2n}$ we define

$$F_{a,b,c}(p) := \mu(x_1) \cdots \mu(x_n) \mu(y_1) \cdots \mu(y_n).$$

Set $U = (-(a + b), (a + b))$ and let $\varphi = \varphi_{a,b,c}$ be the time-one map of $F_{a,b,c}$. Then we have the following

Theorem 1.3. *Let (N, ω) be a $2n$-dimensional integral closed symplectic manifold, and let $N = U_1 \cup \ldots \cup U_r$, where each U_i is a canonical chart domain, $U_1 = U$ and $\mathrm{vol}(U_i) \leq \mathrm{vol}(U)$ for $i = 1, \ldots, r$. Then*

$$\frac{a^{2n} c^{2n}}{(a+b)^{2n}(2n+2)r} \leq \varrho_H(\mathrm{id}, \varphi),$$

where φ is viewed as belonging to $\mathrm{Ham}(N, \omega)$.

Another version of this theorem encompasses exact open symplectic manifolds as well.

Theorem 1.4. *Let (N, ω) be either integral closed or exact open. Under the above notation*

$$\frac{a^{2n} c^{2n}}{(a+b)^{2n}(4n^2 + 6n + 2)} \leq \varrho_H(\mathrm{id}, \varphi).$$

In the local case this inequality may be better, see (7.17). Observe that, in general, the above inequalities are not a consequence of the energy-capacity inequality. In fact, we can take c large enough without changing the support of φ.

Recall that a group G is called *unbounded* if it carries a bi-invariant metric which is unbounded. Otherwise it is called *bounded*. Then we have

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Corollary 1.5. *Let (M, α) and (N, ω) be as in Claim 1.2. Then the metrics ϱ_α on $\mathrm{Cont}_0(M, \alpha)$ and ϱ_H on $\mathrm{Ham}(N, \omega)$ are unbounded. Consequently, the groups $\mathrm{Cont}_0(M, \alpha)$ and $\mathrm{Ham}(N, \omega)$ are unbounded.*

Indeed, it suffices to take c tending to ∞.

There are some results related to Corollary 1.5. For instance, Sikorav proved in [19] that the Hofer metric on the group $\mathrm{Ham}(\mathbb{R}^{2n}, \omega_{st})$ is unbounded, but not stably unbounded. The unboundedness of ϱ_H is established for surfaces, complex projective spaces with the Fubini-Study symplectic form and closed manifolds with $\pi_2 = 0$, see [3] and references therein. Notice that contrary to Corollary 1.5 the identity components of most of diffeomorphism groups are bounded (see [3], [18]), but it is still not known if it is the case of all manifolds. In the contact category several result concerning the (un)boundedness of $\mathrm{Cont}_0(M, \xi)$ and its universal covering group have recently been proved by Colin and Sandon [4], and Fraser, Polterovich and Rosen [7].

2. The Hofer metric

First we recall the definition of the Hofer metric, c.f. [9], [12]. Let (M, ω) be a symplectic manifold. Recall that there is a one-to-one correspondence between isotopies $\{f_t\}$ in $\mathrm{Ham}(M, \omega)$ and smooth families of functions H_t (up to constant) in $\mathrm{C}_c^\infty(M, \mathbb{R})$ given by

$$i_{X_t}\omega = \mathrm{d}\,H_t,$$

where $X_t = \dot{f}_t$ is defined by $\frac{\partial f_t}{\partial t} \circ f_t^{-1}$. Let $\mathfrak{ham}(M, \omega)$ be the Lie algebra of $\mathrm{Ham}(M, \omega)$, i.e. the totality of compactly supported Hamiltonian vector fields on (M, ω). Then for $X \in \mathfrak{ham}(M, \omega)$ we set

$$(2.1) \qquad \parallel X \parallel_\infty = \parallel H_X \parallel_\infty = \max_{p \in M} H_X(p) - \min_{p \in M} H_X(p),$$

where $H_X \in \mathrm{C}_c^\infty(M, \mathbb{R})$ such that $i_X\omega = \mathrm{d}\,H_X$ ($\parallel \cdot \parallel_\infty$ is a norm on $\mathfrak{ham}(M, \omega)$ even if M compact). Observe that

$$(2.2) \qquad \parallel H \circ \varphi \parallel_\infty = \parallel H \parallel_\infty$$

for all $H \in \mathrm{C}_c^\infty(M, \mathbb{R})$ and $\varphi \in \mathrm{Ham}(M, \omega)$, that is the norm $\parallel \cdot \parallel_\infty$ is invariant w.r.t. the adjoint action of $\mathrm{Ham}(M, \omega)$ on $\mathfrak{ham}(M, \omega)$. Now the *Hofer length*

of a Hamiltonian isotopy $\{f_t\}$ is defined as

$$(2.3) \qquad\qquad l_H(\{f_t\}) = \int_0^1 \| \dot{f}_t \|_\infty \, dt.$$

The *Hofer norm* (or the *Hofer energy*) is then defined for $\varphi \in \text{Ham}(M, \omega)$ by

$$(2.4) \qquad\qquad E_H(\varphi) = \inf_{\{f_t\}} (l_H(\{f_t\})),$$

where $\{f_t\}$ runs over all Hamiltonian isotopies joining id to φ.

Let G be a group. For a function $\nu : G \to [0, \infty)$ such that $\nu(e) = 0$ consider the following conditions. For any $g, h \in G$

   (1) $\nu(g^{-1}) = \nu(g)$;
   (2) $\nu(gh) \le \nu(g) + \nu(h)$;
   (3) $\nu(g) > 0$ if and only if $g \ne e$;
   (4) $\nu(hgh^{-1}) = \nu(g)$.

Then ν is called a *pseudo-norm* (resp. *norm*) if (1)-(2) (resp. (1)-(3)) are fulfilled. If (3) is not satisfied, ν is called *degenerate*. Next, ν is *conjugation-invariant* if (4) is satisfied. In view of Lemma 3.1 below and (2.2) it is easily seen that E_H is a conjugation-invariant pseudo-norm.

The following theorem was proved by Hofer in [8] for $M = \mathbb{R}^{2n}$. It was generalized for some other symplectic manifolds by Polterovich in [15]. Finally, the proof for all symplectic manifolds was given by Lalonde and McDuff in [11]. In all three proofs the Hofer's idea of displacement energy and hard symplectic methods are in use.

Theorem 2.1. $E_H : \text{Ham}(M, \omega) \to [0, \infty)$ *is a norm. Consequently,* $\varrho_H(\varphi, \psi) := E_H(\varphi\psi^{-1})$ *is a bi-invariant metric, called the* Hofer *metric.*

The Hofer metric plays a crucial role in symplectic topology and various important notions and facts are expressed in terms of it (see, e.g., [9], [12], [13], [16]). The original proof of its non-degeneracy for $M = \mathbb{R}^{2n}$ is based on the action principle and a crucial role in it is played by the action spectrum. The Hofer metric is intimately related, on the one hand, to a capacity c_0 (c.f. [9]) and hence to periodic orbits, and on the other hand to the displacement energy.

3. The groups of contactomorphisms

Let $f \in \text{Diff}_0(M)$ and $\{f_t\}, t \in \mathbb{R}$, be a compactly supported smooth isotopy such that $f_1 = f, f_0 = \text{id}$. Then $\{f_t\}$ determines a smooth family of vector fields $\{\dot{f}_t\}$ in $\mathfrak{X}_c(M)$, the Lie algebra of all compactly supported vector fields on M. Namely for all $p \in M$ and $t \in \mathbb{R}$ we have

$$(3.1) \qquad\qquad \dot{f}_t(p) = \frac{\partial f_t}{\partial t}(f_t^{-1}(p)).$$

Moreover, there is the one-to-one correspondence between

$$\{f_t\} \in \mathrm{C}_e^\infty(\mathbb{R}, \mathrm{Diff}_0(M)) = \{f : \mathbb{R} \to \mathrm{Diff}_0(M)|\, f(0) = \mathrm{id}\}$$

and $\{\dot{f}_t\} \in \mathrm{C}^\infty(\mathbb{R}, \mathfrak{X}_c(M))$, see, e.g., [10] for more details. In particular, a time-independent vector field $X \in \mathfrak{X}_c(M)$ corresponds to its flow $\mathrm{Fl}^X \in \mathrm{C}_e^\infty(\mathbb{R}, \mathrm{Diff}_0(M))$.

Likewise, for any $\varphi \in \mathrm{Diff}_0(M)$ the space $\mathrm{C}_\varphi^\infty(\mathbb{R}, \mathrm{Diff}_0(M)) = \{f : \mathbb{R} \to \mathrm{Diff}_0(M)|\, f(0) = \varphi\}$ identifies with $\mathrm{C}^\infty(\mathbb{R}, \mathfrak{X}_c(M))$ by $\{f_t\} \mapsto \{\overbrace{\dot{f_t \circ \varphi^{-1}}}\}$.

The following is easy to check.

Lemma 3.1. *Let* $\{f_t\}, \{g_t\} \in \mathrm{C}_e^\infty(\mathbb{R}, \mathrm{Diff}_0(M))$ *and* $\phi \in \mathrm{Diff}(M)$. *Then:*

(1) $\overbrace{\dot{f_t g_t}} = \dot{f}_t + (f_t)_*(\dot{g}_t)$.

(2) $\overbrace{\dot{f_t^{-1}}} = -(f_t^{-1})_*(\dot{f}_t)$.

(3) $\overbrace{\dot{\phi f_t \phi^{-1}}} = \phi_*(\dot{f}_t)$.

Let $(M, \xi = \ker(\alpha))$ be a co-oriented contact manifold with $\dim(M) = 2n+1$. The contact form α can be put into the following normal form. For any $p \in M$ there is a chart $(x_1, \ldots, x_n, y_1, \ldots, y_n, z) : M \supset U \to u(U) \subset \mathbb{R}^{2n+1}$, centered at p, such that $\alpha|_U = \mathrm{d}\,z - y_1\,\mathrm{d}\,x_1 - \ldots - y_n\,\mathrm{d}\,x_n$.

The symbol $\mathfrak{cont}(M, \xi)$ will stand for the Lie algebra of all contact vector fields, i.e. $X \in \mathfrak{cont}(M, \xi)$ iff $L_X \alpha = \mu_X \alpha$ for some function $\mu_X \in \mathrm{C}^\infty(M, \mathbb{R})$, where L is the Lie derivative. This definition is independent of α, though μ_X depends on α. Let $\mathfrak{cont}_c(M, \xi)$ be the Lie subalgebra of compactly supported elements of $\mathfrak{cont}(M, \xi)$. Then $\mathfrak{cont}_c(M, \xi)$ is the Lie algebra of the Lie group $\mathrm{Cont}_0(M, \xi)$ (c.f. [10]) and in view of (3.1) we get the bijection

$$\mathrm{C}_e^\infty(\mathbb{R}, \mathrm{Cont}_0(M, \xi)) \ni \{f_t\} \mapsto \{\dot{f}_t\} \in \mathrm{C}^\infty(\mathbb{R}, \mathfrak{cont}_c(M, \xi)).$$

Set $I = [0, 1]$. For a Lie group G and $g, h \in G$ we introduce the notation

$$\mathcal{J}_g G = \{f \in \mathrm{C}^\infty(I, G)|f(0) = g\}, \quad \mathcal{J}_g^h G = \{f \in \mathrm{C}^\infty(I, G)|f(0) = g, f(1) = h\}$$

for the isotopy groups of G.

Then the above bijection induces the bijection for I (as well as for any interval of $\mathbb{R}$ with some initial condition fixed)

(3.2) $$\mathcal{J}_{\mathrm{id}} \mathrm{Cont}_0(M, \xi) \to \mathrm{C}^\infty(I, \mathfrak{cont}_c(M, \xi)).$$

In particular, one has $L_{X_t}\alpha = \mu_{X_t}\alpha$ with $\mu_{X_t} = (\partial \ln \lambda_{f_t}/\partial t)f_t^{-1}$ where $f_t^*\alpha = \lambda_{f_t}\alpha$ and $X_t = \dot{f}_t$. Next we define $\mathfrak{cont}_c(M, \alpha) = \{X \in \mathfrak{cont}_c(M, \xi) : \mu_X = 0\}$, and the bijection

$$\mathcal{J}_{\mathrm{id}} \mathrm{Cont}_0(M, \alpha) \to \mathrm{C}^\infty(I, \mathfrak{cont}_c(M, \alpha)).$$

Let R_α denote the unique vector field satisfying $i_{R_\alpha}\alpha = 1$ and $i_{R_\alpha}\,\mathrm{d}\,\alpha = 0$, where i_X is the interior product w.r.t. X. R_α is called the *Reeb vector field*. Clearly $R_\alpha \in \mathfrak{cont}(M,\alpha)$. A vector field X is called *horizontal* if $i_X\alpha = 0$. A dual concept is a *semibasic* form, i.e. any 1-form γ such that $\gamma(R_\alpha) = 0$. We have the $\mathbb{R}$-linear isomorphism

$$I_{d\alpha} : \mathfrak{X}(M) \ni X \mapsto i_X\,\mathrm{d}\,\alpha + \alpha(X)\alpha \in \mathfrak{F}^1(M)$$

between the space of vector fields $\mathfrak{X}(M)$ and the space of 1-forms $\mathfrak{F}^1(M)$. The isomorphism $I_{d\alpha}$ preserves the duality, that is

(3.3) $\qquad X \in \mathfrak{X}(M)$ is horizontal $\quad\Longleftrightarrow\quad I_{d\alpha}(X)$ is semi-basic.

Given any function $H \in \mathrm{C}^\infty(M,\mathbb{R})$, we get a semi-basic 1-form

$$\theta_H = \mathrm{d}\,H(R_\alpha)\alpha - \mathrm{d}\,H.$$

It defines uniquely a horizontal vector field Y_H such that

(3.4) $\qquad\qquad\qquad\qquad i_{Y_H}\,\mathrm{d}\,\alpha = \mathrm{d}\,H(R_\alpha)\alpha - \mathrm{d}\,H.$

As a consequence we have the existence of the following isomorphism I_α, an important tool in contact geometry.

Proposition 3.2. *There is an $\mathbb{R}$-linear isomorphism*

$$I_\alpha : \mathfrak{cont}(M,\xi) \ni X \mapsto i_X\alpha = \alpha(X) \in \mathrm{C}^\infty(M,\mathbb{R}).$$

In particular, $\mathrm{C}_c^\infty(M,\mathbb{R})$ is identified with $\mathfrak{cont}_c(M,\xi)$ by means of α.

For $H \in \mathrm{C}^\infty(M,\mathbb{R})$ we have

(3.5) $\qquad I_\alpha^{-1}(H) = HR_\alpha + I_{d\alpha}^{-1}((i_{R_\alpha}\,\mathrm{d}\,H)\alpha - \mathrm{d}\,H) = HR_\alpha + Y_H.$

Set $X_H = I_\alpha^{-1}(H)$. Notice that, in view of (3.3), (3.4) and the Cartan formula,

(3.6) $\qquad\qquad\qquad\qquad L_{X_H}\alpha = (\mathrm{d}\,H(R_\alpha))\alpha,$

that is X_H is indeed an element of $\mathfrak{cont}(M,\xi)$. It follows that any *basic* function $H \in \mathrm{C}^\infty(M,\mathbb{R})$, i.e. a function invariant under the Reeb flow, gives rise to a horizontal vector field Y_H given by

(3.7) $\qquad\qquad\qquad\qquad i_{Y_H}\,\mathrm{d}\,\alpha = -\,\mathrm{d}\,H.$

In view of (3.2) and Prop. 3.2 we get the bijective correspondence

(3.8) $\qquad \Psi_\alpha : \mathcal{J}_\mathrm{id}\,\mathrm{Cont}_0(M,\xi) \ni \{f_t\} \mapsto \Psi_\alpha(\{f_t\}) \in \mathrm{C}_c^\infty(I \times M,\mathbb{R}).$

Furthermore, denote by $\mathrm{C}_b^\infty(M,\mathbb{R})$ the space of compactly supported basic functions on M. Hence in view of (3.6) there is a one-to-one correspondence

(3.9) $\qquad \Psi_\alpha : \mathcal{J}_\mathrm{id}\,\mathrm{Cont}_0(M,\alpha) \ni \{f_t\} \mapsto \Psi_\alpha(\{f_t\}) \in \mathrm{C}_b^\infty(I \times M,\mathbb{R}).$

Now from Lemma 3.1 we derive the following

Lemma 3.3. *Let Ψ_α be as in (3.8), $\{f_t\}, \{g_t\} \in \mathcal{J}_{\mathrm{id}}\,\mathrm{Cont}_0(M,\xi)$ and $\varphi \in \mathrm{Cont}(M,\xi)$. Then:*

(1) $\Psi_\alpha(\{f_t g_t\}) = \Psi_\alpha(\{f_t\}) + (\lambda_{f_t} \cdot \Psi_\alpha(\{g_t\})) \circ f_t^{-1}$;

(2) $\Psi_\alpha(\{f_t^{-1}\}) = -\lambda_{f_t}^{-1} \cdot (\Psi_\alpha(\{f_t\}) \circ f_t)$;

(3) $\Psi_\alpha(\{f_t^{-1} g_t\}) = \lambda_{f_t}^{-1} \cdot \left((\Psi_\alpha(\{g_t\}) - \Psi_\alpha(\{f_t\})) \circ f_t \right)$;

(4) $\Psi_\alpha(\{\varphi f_t \varphi^{-1}\}) = \lambda_\varphi \cdot (\Psi_\alpha(\{f_t\}) \circ \varphi^{-1})$.

We will deal with the standard contact form $\alpha_{st} = \mathrm{d}\,z - \sum_{i=1}^{n} y_i \,\mathrm{d}\,x_i$ on $\mathbb{R}^{2n+1}$ or $\mathbb{R}^{2n} \times \mathbb{S}^1$. Then we have $X_{\alpha_{st}} = \frac{\partial}{\partial z}$ and $\mathrm{d}\,\alpha_{st} = \sum_{i=1}^{n} \mathrm{d}\,x_i \wedge \mathrm{d}\,y_i$. Observe that $\xi_{st} = \ker(\alpha_{st})$ is generated by $Y_i = \frac{\partial}{\partial y_i}$ and $X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}$, where $i = 1, \ldots, n$.

Next it is easily seen that $I_{d\alpha_{st}}(Y_i) = -\mathrm{d}\,x_i$ and $I_{d\alpha_{st}}(X_i) = \mathrm{d}\,y_i$. For every $X \in \mathfrak{X}(\mathbb{R}^{2n+1}, \alpha_{st})$ we have $i_X \alpha_{st} \in \mathrm{C}^\infty(\mathbb{R}^{2n+1}, \mathbb{R})$ by

$$(3.10) \qquad i_X \alpha_{st} = u_0 - \sum_{i=1}^{n} y_i u_i \quad \text{if} \quad X = u_0 \frac{\partial}{\partial z} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i} + u_{n+i} \frac{\partial}{\partial y_i}.$$

Conversely, in view of (3.5) and the above equalities, we have

$$(3.11) \;\; X_H = \left(H - \sum_{i=1}^{n} y_i \frac{\partial H}{\partial y_i} \right) \frac{\partial}{\partial z} - \sum_{i=1}^{n} \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} + y_i \frac{\partial H}{\partial z} \right) \frac{\partial}{\partial y_i},$$

for all $H \in \mathrm{C}^\infty(\mathbb{R}^{2n+1}, \mathbb{R})$. Indeed, it is easily checked that $\alpha(X_H) = H$ and $L_{X_H}\alpha = \frac{\partial H}{\partial z}\alpha$. These equalities imply (3.11).

Throughout we will write x instead of $(x_1, \ldots, x_n)$ and y instead of $(y_1, \ldots, y_n)$. We specify some elements in $\mathrm{Cont}_0(\mathbb{R}^{2n} \times \mathbb{S}^1, \alpha_{st})$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$:

(1) The translations $\tau_t^{(z)} : (x, y, z) \to (x, y, z + t)$ for $t \in \mathbb{R}$. Here H is the constant function 1 and $X_{H_0} = R_{\alpha_{st}}$.

(2) The translations $\tau_t^{(x_i)} : (x, y, z) \to (x + t\mathbf{1}_i, y, z)$, $i = 1, \ldots, n$ and $t \in \mathbb{R}$, with $H(x, y, z) = -y_i$ and $X_H = \frac{\partial}{\partial x_i}$.

(3) The pseudo-translations $\tau_t^{(y_i)} : (x, y, z) \to (x, y + t\mathbf{1}_i, z + tx_i)$ for $i = 1, \ldots, n$ and $t \in \mathbb{R}$. Here $H(x, y, z) = x_i$ and $X_H = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}$.

Observe that the maps $\tau_t^{(y_i)}$ from (3) cannot be "replaced" by the translations along the y_i axes, since such translations are not contactomorphisms as they do not preserve the contact distribution ξ.

Proposition 3.4. (1) *If a diffeomorphism f of $\mathbb{R}^{2n} \times \mathbb{S}^1$ is a contactomorphism, where $f = (f_1, \ldots, f_{2n}, f_{2n+1})$ w.r.t. the coordinates (x, y, z), then we have*

$$\frac{\partial f_{2n+1}}{\partial z} - \sum_{j=1}^{n} f_{n+j} \frac{\partial f_j}{\partial z} = \lambda_f.$$

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In particular, if f is independent of z then $\lambda_f = 1$ on $\mathbb{R}^{2n} \times \mathbb{S}^1$.

(2) Given a contact manifold (M, α), if there is a nonconstant $H \in \mathrm{C}_c^\infty(M, \mathbb{R})$ independent of z, then the group $\mathrm{Cont}(M, \alpha)$ is non-trivial.

Proof. (1) follows from the coordinate expression for f, see Prop. 2.2 in [17]. Now, (2) is a consequence of (1) and (3.7). Namely, take elements of the flow Fl^{Y_H}, where H is independent of z. $\qquad\square$

Let $\pi : (M, \alpha) \to (N, \omega)$ be a prequantization bundle. For a subset $C \subset N$ let $\mathrm{Cont}_C(M, \alpha)$ stand for the totality of $f \in \mathrm{Cont}_0(M, \alpha)$ such that there exists an isotopy $\{f_t\}$ compactly supported in $C \times \mathbb{S}^1$ with $f_0 = \mathrm{id}$ and $f_1 = f$. In the sequel we shall need the fragmentation lemma.

Lemma 3.5. *Let (M, α) and (N, ω) satisfy the assumption of Claim 1.2. Suppose $U_1, \ldots, U_r$ is an open covering of N, i.e. $N = U_1 \cup \cdots \cup U_r$, where possibly $U_i = \bigsqcup_k U_{i,k}$ such that $\{U_{i,k}\}_{k=1}^\infty$ is a family of pairwise disjoint, locally finite sets for all i. Moreover, the bundle π trivializes over each $U_{i,k}$. Then for every compact subset $C \subset N$ there is a C^1 neighborhood $\mathcal{U}$ of id in $\mathrm{Cont}_C(M, \alpha)$ and smooth maps preserving the identity*

$$P_i : \mathcal{U} \to \mathrm{Cont}_C(M, \alpha), \quad i = 1, \ldots, r,$$

such that

$$\mathrm{supp}(P_i(f)) \subset U_i \times \mathbb{S}^1, \quad f = P_r(f) \circ \ldots \circ P_1(f)$$

for any $f \in \mathcal{U}$.

The proof follows from Prop. 3.2 and is well-known (e.g. [17]).

4. The pseudo-metric ϱ_α

Let $(M, \xi = \ker(\alpha))$ be a (co-oriented) contact manifold. We shall deal with the pseudo-metric ϱ_α on $\mathrm{Cont}_0(M, \alpha)$.

In view of Prop. 3.2 it is obvious that $\| \cdot \|_\alpha$ given by

$$(4.1) \qquad \qquad \| X \|_\alpha = \| \alpha(X) \|_\infty$$

is a pseudo-norm on $\mathfrak{cont}_c(M, \xi)$. If M is open then for $X \in \mathfrak{cont}_c(M, \xi)$ we have $\max_{p \in M} |\alpha(X)| \leq \| X \|_\alpha$. That is, $\| \cdot \|_\alpha$ is a norm for M open. Contrary to the Hamiltonian case, $\| \cdot \|_\alpha$ is not a norm if M is closed. In fact, the Reeb vector field R_α belongs to $\mathfrak{cont}_c(M, \alpha)$ and clearly $\| R_\alpha \|_\alpha = 0$. Next ϱ_α possesses the invariance property, similar to (2.2).

Proposition 4.1. *For any $X \in \mathfrak{cont}(M, \xi)$ and $\varphi \in \mathrm{Cont}(M, \alpha)$ we have $\| \mathrm{Ad}_\varphi(X) \|_\alpha = \| X \|_\alpha$.*

Proof. It follows from the equality $\alpha(\varphi_* X) = \varphi^* \alpha(X) \circ \varphi^{-1} = (\lambda_\varphi \alpha(X)) \circ \varphi^{-1} = \alpha(X) \circ \varphi^{-1}$ for all $X \in \mathfrak{X}(M)$ and $\varphi \in \mathrm{Cont}(M, \alpha)$. $\qquad\square$

For a contact isotopy $\{f_t\}$, $t \in I$, in $\mathrm{Cont}_0(M, \xi)$ we introduce the notion of *contact length* by

$$(4.2) \qquad l_\alpha(\{f_t\}) = \int_0^1 \| \dot{f}_t \|_\alpha \, dt = \int_0^1 \| F_t \|_\infty \, dt,$$

where $F \in C_c^\infty(I \times M, \mathbb{R})$ corresponds to $\{f_t\}$ and $F_t = F(t, \cdot)$. Here we do not assume that $f_0 = \mathrm{id}$ and we have the equality

$$(4.3) \qquad l_\alpha(\{f_{1-t}\}) = l_\alpha(\{f_t\}),$$

and the right-invariance of l_α

$$(4.4) \qquad \forall \varphi \in \mathrm{Cont}_0(M, \xi), \quad l_\alpha(\{f_t \varphi\}) = l_\alpha(\{f_t\}).$$

Lemma 4.2. *If $\sigma : [a, b] \to I$ be a smooth non-decreasing surjection then for any isotopy $\{f_t\}$ in $\mathrm{Cont}_0(M, \xi)$ one has $l_\alpha(\{f_{\sigma(t)}\}_{a \le t \le b}) = l_\alpha(\{f_t\})$.*

Proof. Suppose $F \in C_c^\infty(I \times M, \mathbb{R})$ corresponds by (3.8) to $\{f_t\}$ and $F^\sigma \in C_c^\infty(I \times M, \mathbb{R})$ corresponds to $\{f_{\sigma(t)}\}$. Then we have

$$F^\sigma(t, x) = \alpha_x(\overbrace{\dot{f_{\sigma(t)}}}(x)) = \alpha_x \left(\sigma'(t) \frac{\partial f_\tau}{\partial \tau} \Big|_{\tau = \sigma(t)} (f_{\sigma(t)}^{-1}(x)) \right)$$

$$= \sigma'(t) \alpha_x(\dot{f}_{\sigma(t)}(x)) = \sigma'(t) F(\sigma(t), x).$$

It follows that

$$l_\alpha(\{f_{\sigma(t)}\}) = \int_a^b \| F_t^\sigma \|_\infty \, dt = \int_a^b \sigma'(t) \| F_{\sigma(t)} \|_\infty \, dt$$

$$= \int_0^1 \| F_t \|_\infty \, dt = l_\alpha(\{f_t\}),$$

as required. $\qquad \square$

Note that for any $0 < s < 1$ one has

$$(4.5) \qquad l_\alpha(\{f_t\}_{t \in I}) = l_\alpha(\{f_t\}_{0 \le t \le s}) + l_\alpha(\{f_t\}_{s \le t \le 1}).$$

As a consequence of the above facts we get

Proposition 4.3. *For all isotopies $\{f_t\}$, $\{g_t\}$ in $\mathrm{Cont}_0(M, \xi)$ (resp. $\mathrm{Cont}_0(M, \alpha)$) we have:*

(1) *For any $0 < \delta < \frac{1}{2}$ there exists an isotopy $\{\bar{f}_t\}$ in $\mathrm{Cont}_0(M, \xi)$ (resp. $\mathrm{Cont}_0(M, \alpha)$) with $l_\alpha(\{\bar{f}_t\}) = l_\alpha(\{f_t\})$ such that $f_t = f_0$ for $|t| \le \delta$ and $f_t = f_1$ for $|1 - t| \le \delta$.*

(2) *There is $\{\tilde{f}_t\}$ in $\mathrm{Cont}_0(M, \xi)$ (resp. $\mathrm{Cont}_0(M, \alpha)$) such that $\Psi_\alpha(\{\tilde{f}_t\})$ is 1-periodic in t, $\tilde{f}_0 = f_0$, $\tilde{f}_1 = f_1$, and $l_\alpha(\{\tilde{f}_t\}) = l_\alpha(\{f_t\})$.*

(3) *$l_\alpha(\{\bar{f}_t\} \star \{\bar{g}_t\}) = l_\alpha(\{f_t\}) + l_\alpha(\{g_t\})$, where $\{\bar{f}_t\}$, $\{\bar{g}_t\}$ are as in (1), and $\{\bar{f}_t\} \star \{\bar{g}_t\}$ is their concatenation.*

(4) *$\forall \varphi \in \mathrm{Cont}(M, \alpha), \quad l_\alpha(\{\varphi f_t \varphi^{-1}\}) = l_\alpha(\{f_t\})$.*

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Proof. In fact, (1) is a consequence of Lemma 4.2, and (2) and (3) follow from (1) and (4.5). Prop. 4.1 and (4.4) yield (4). $\qquad\square$

For $\varphi, \psi \in \mathrm{Cont}_0(M, \alpha)$ define

(4.6) $$\varrho_\alpha(\varphi, \psi) = \inf_{\{f_t\}} \{l_\alpha(\{f_t\})\},$$

where $\{f_t\}$ is an element of $\mathcal{J}_\varphi^\psi \mathrm{Cont}_0(M, \alpha)$. Then, in view of Prop.4.3, (4.3) and (4.4) ϱ_α is a bi-invariant pseudo-metric on $\mathrm{Cont}_0(M, \alpha)$.

In order to complete the proof of Theorem 1.1 we shall need the energy-capacity inequality in the contact case.

Theorem 4.4. *(Theorem 1.1 in* [14]*) Let $(M, \xi = \ker(\alpha))$ is a contact manifold. Suppose that $\phi \in \mathrm{Cont}_0(M, \xi)$ is the time-one map of $H \in \mathrm{C}_c^\infty(I \times M, \mathbb{R})$ and that ϕ displaces a ball. Then there exists a constant $C > 0$, independent of the contact Hamiltonian H, such that*

$$0 < Ce^{-|h|} \le \| H \| := \max_{p \in M} |H(p)|,$$

where $h \in \mathrm{C}_c^\infty(I \times M, \mathbb{R})$ is the conformal factor of the isotopy $\{\phi_t\} = \Psi_\alpha^{-1}(H)$, i.e. $\phi_t^* \alpha = e^{h(t, \cdot)} \alpha$.

Proof of Theorem 1.1 We already know that ϱ_α is a bi-invariant pseudo-metric on $\mathrm{Cont}_0(M, \alpha)$. For M open the non-degeneracy follows from Theorem 4.4 since in the group $\mathrm{Cont}_0(M, \alpha)$ $h(t, \cdot) = 0$ on M and from the inequality $\| H \| \le \| H \|_\infty$. For M closed the elements of the flow of R_α belong to $\mathrm{Cont}_0(M, \alpha)$ and $\| R_\alpha \|_\alpha = 0$ for all t.

5. RELATION TO BANYAGA-DONATO METRIC

Let us remark that Banyaga and Donato introduced in [2] a bi-invariant metric on $\mathrm{Cont}_0(M, \alpha)$, where (M, α) is a compact regular contact manifold satisfying some additional condition, by using the Hofer metric on the corresponding Hamiltonian group of the base of M. Namely, if (M, α) is as above then the mapping $c : \mathrm{Cont}_0(M, \alpha) \to \mathbb{R}$ given by

$$c(f_t) = \frac{1}{\mathrm{vol}(M)} \int_M \left(\int_0^1 F_t(p) \, \mathrm{d}t \right) \nu_\alpha$$

induces an epimorphism $\tilde{c} : \widetilde{\mathrm{Cont}(M, \alpha)} \to \mathbb{R}$, where $F = \Psi_\alpha(\{f_t\})$ corresponds to $\{f_t\}$ and $\widetilde{\mathrm{Cont}(M, \alpha)}$ is the universal covering of $\mathrm{Cont}(M, \alpha)$. Moreover, it is assumed in [2] that

$$\tilde{c}(\pi_1(\mathrm{Cont}(M, \alpha))) = \mathbb{Z}.$$

Now the contact length defined by Banyaga and Donato takes the form

$$(5.1) \qquad l_{BD}(\{f_t\}) = |c(f_t)| + \int_0^1 \parallel \dot{f}_t \parallel_\infty dt,$$

where $\{f_t\}$ is an isotopy in $\mathrm{Cont}_0(M,\alpha)$. Next, the bi-invariant metric ϱ_{BD} is defined by (4.6), where l_α is replaced by l_{BD}.

Recently, Müller and Spaeth introduce in [14] the definition of the Banyaga-Donato metric for all contact manifolds (Theorem 1.2 in [14]) by using a length slightly different than l_{BD} given by (5.1). Namely, for $F = \Psi_\alpha(\{f_t\})$ they defined

$$(5.2) \qquad l'_{BD}(\{f_t\}) = \int_0^1 (\parallel \dot{f}_t \parallel_\infty + |c(F_t)|) dt,$$

where c is the average value of a function, that is $c(F) := \int_M F\nu_\alpha$. However we have that the metrics ϱ_{BD} defined by (5.1) and ϱ'_{BD} defined by (5.2) coincide in view of Lemma 10.3, [14].

The non-degeneracy of ϱ_{BD} was proven in [14] for all contact manifolds by applying the energy-capacity inequality for contact structures (Theorem 4.4). This inequality is obtained by means of the energy-capacity inequality in the symplectization of the contact manifold in question.

Since for M open and for all $F \in \mathrm{C}_c^\infty(M,\mathbb{R})$ we have $\parallel F \parallel \leq \parallel F \parallel_\infty + |c(F)| \leq 3 \parallel F \parallel$, where $\parallel F \parallel = \max_{p \in M} |F(p)|$, the metrics ϱ_α and ϱ_{BD} are then equivalent. On the other hand, it is very likely that these metrics are different (though difficult to show).

In the sequel we shall not appeal to the metric ϱ_{BD}.

6. RELATION TO HOFER METRIC

In this section we shall see how the pseudo-metric ϱ_α is related to the Hofer metric ϱ_H.

Set $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Now let us consider the assumption of Claim 1.2. Let $\pi : M \to N$ be a principal $\mathbb{S}^1$-bundle with the $\mathbb{S}^1$-action on M generated by the flow of a vector field R with period 1. Suppose that α is a connection 1-form on this bundle, i.e. α fulfills $L_R\alpha = 0$ and $i_R\alpha = 1$. In view of the Cartan formula we get $L_R(\mathrm{d}\alpha) = 0$ and $i_R\mathrm{d}\alpha = 0$. Consequently, $\mathrm{d}\alpha$ determines a 2-form ω on N such that $\pi^*\omega = \mathrm{d}\alpha$. Since π^* is injective, it follows from $\pi^*\mathrm{d}\omega = \mathrm{d}^2\alpha = 0$ so that $\mathrm{d}\omega = 0$. Then ω is called the curvature form of α.

The above situation is well-known and occurs in the following cases:

(1) Due to the first part of Boothby-Wang theorem (see, e.g., [6], 7.2.4), if (N,ω) is a closed symplectic manifold with symplectic integral form ω and $\pi : M \to N$ is the principal $\mathbb{S}^1$-bundle with Euler class $-[\omega] \in H^2(N,\mathbb{Z})$, then there exists a connection form α on M such that α is a regular contact form, the curvature form of α is ω, and the infinitesimal

generator of the $\mathbb{S}^1$-action is R_α. Recall that a nowhere vanishing vector field R on M is called *regular* if each point of M admits a flow box which meets at most once any integral curve of R. Next a contact form is *regular* if so is its Reeb vector field.

More generally, in view of the second part of Boothby-Wang theorem ([6], 7.2.5), if α is a regular contact form on a closed manifold M then, after rescaling it, α is the connection form on a principal $\mathbb{S}^1$-bundle $\pi : M \to N$ over a symplectic manifold (N, ω), with ω the curvature form of α. Then (M, α) is called the prequantization of (N, ω) and $\mathrm{Cont}_0(M, \alpha)$ is the group of quantomorphisms (c.f. Souriau [20]).

(2) If (N, ω) is an open exact symplectic manifold with $\omega = -\mathrm{d}\,\theta$, then we define a contact manifold $(M, \xi = \ker(\alpha))$, where $M = N \times \mathbb{S}^1$, $\alpha = \mathrm{d}\,z - \theta$, and z is the coordinate on $\mathbb{S}^1$.

In each case the contact distribution coincides with the horizontal distribution of the connection in $\pi : M \to N$ determined by the form α.

Next recall the definition of a homomorphism

$$(6.1) \qquad\qquad q : \mathrm{Cont}_0(M, \alpha) \to \mathrm{Ham}(N, \omega).$$

Let $\{f_t\}$ be an isotopy in $\mathrm{Cont}_0(M, \alpha)$ joining id to $f = f_1$ and let $F = \Psi_\alpha(\{f_t\})$ be the corresponding element of $C_b^\infty(I \times M, \mathbb{R})$ by (3.9). Then $L_{\dot{f}_t}\alpha = 0$ and the Cartan formula yields the equalities

$$i_{\dot{f}_t}\,\mathrm{d}\,\alpha = -\mathrm{d}(i_{\dot{f}_t}\alpha) = -\mathrm{d}\,F_t.$$

Since F_t is basic there is a smooth family H_t in $C_c^\infty(N, \mathbb{R})$ given by

$$(6.2) \qquad\qquad -F_t = H_t \circ \pi.$$

In view of (3.7) there is a unique smooth curve of vector fields $Y_t = Y_{F_t}$ such that $\alpha(Y_t) = 0$ and $i_{Y_t}\,\mathrm{d}\,\alpha = -\mathrm{d}\,F_t$, that is Y_t is the horizontal part of $\dot{f}_t$. It is easily seen that Y_t coincides with the horizontal lift of $X_t = X_{H_t}$ w.r.t. the connection α. Consequently, $\{f_t\}$ projects onto $\{h_t\}$. Then we define (6.1) by $q(f) = h_1$. Due to (6.2)

$$(6.3) \qquad\qquad l_H(\{q(f_t)\}) = l_\alpha(\{f_t\}),$$

where l_H is given by (2.3), and

$$(6.4) \qquad \forall f \in \mathrm{Cont}_0(M, \alpha), \quad \varrho_H(\mathrm{id}, q(f)) = \varrho_\alpha(\mathrm{id}, f).$$

Furthermore, in view of (3.5) and Lemma 3.3(1), we get

(6.5)
$$(\forall h \in \mathrm{Ham}(N, \omega)), (\forall f \in q^{-1}(h)), \quad q^{-1}(h) = \{\tau_c \circ f : \tau_c = \mathrm{Fl}_c^{R_\alpha}, c \in \mathbb{R}\}.$$

It follows that for (N, ω) open q is actually an isomorphism. In the case of (N, ω) closed we have the exact sequence of groups

$$\{1\} \to \mathbb{S}^1 \to \mathrm{Cont}_0(M, \alpha) \xrightarrow{q} \mathrm{Ham}(N, \omega) \to \{1\}.$$

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Then q induces the epimorphism

(6.6) $\qquad \tilde{q} : \mathcal{J} \operatorname{Cont}_0(M, \alpha) \to \mathcal{J} \operatorname{Ham}(N, \omega), \quad \{f_t\} \mapsto \{q(f_t)\}.$

Due to (6.5) the kernel of $\tilde{q}$ coincides with $C_e^\infty(I, \mathbb{S}^1) \subset C_e^\infty(I, \operatorname{Cont}_0(M, \alpha))$.

The following notion of *contact length* will be also in use. For a strict contact isotopy $\{f_t\}$ in $\operatorname{Cont}_0(M, \alpha)$ we define

(6.7) $\qquad l_\alpha^\mu(\{f_t\}) = \max_{t \in I} \| \dot{f}_t \|_\alpha = \max_{t \in I} \| F_t \|_\infty,$

where $F = \Psi_\alpha(\{f_t\}) \in C_b^\infty(I \times M, \mathbb{R})$. By definitions (4.2) and (6.7) we get

(6.8) $\qquad\qquad\qquad\qquad l_\alpha \leq l_\alpha^\mu.$

However, the length l_α^μ leads to the same pseudo-metric as the length l_α in view of the following

Lemma 6.1. *Let (M, α) satisfy the assumption of Claim 1.2. For any $\phi \in \operatorname{Cont}_0(M, \alpha)$ we have $\varrho_\alpha(\operatorname{id}, \phi) = \inf(l_\alpha^\mu(\{f_t\}))$, where $\{f_t\} \in \mathcal{J}_{\operatorname{id}}^\phi \operatorname{Cont}_0(M, \alpha)$.*

Proof. For an isotopy $\{h_t\}$ in $\operatorname{Ham}(N, \omega)$ we define $l_H^\mu(\{h_t\}) = \max_{t \in I} \| \dot{h}_t \|_\infty$. In view Lemma 5.1.C in [16] and (6.4) we have

$$\varrho_\alpha(\operatorname{id}, \phi) = \varrho_H(\operatorname{id}, q(\phi)) = \inf\{l_H^\mu(\{h_t\})\},$$

where $\{h_t\} \in \mathcal{J}_{\operatorname{id}}^{q(\phi)} \operatorname{Ham}(N, \omega)$. The assertion follows by the surjectivity of (6.6) and the equality $l_H^\mu(\{q(f_t)\}) = l_\alpha^\mu(\{f_t\})$. $\qquad\square$

We shall use the classical simplicity theorem on the symplectomorphism group, due to Banyaga ([1]). This theorem may be formulated as follows.

Theorem 6.2. [1] *Let (N, ω) be a symplectic manifold (without boundary) and let $\operatorname{Ham}(N, \omega)$ be the group of its compactly supported Hamiltonian symplectomorphisms.*

    (1) *If N is closed then the group $\operatorname{Ham}(N, \omega)$ is simple.*
    (2) *If N is open then the commutator subgroup $[\operatorname{Ham}(N, \omega), \operatorname{Ham}(N, \omega)]$ is simple.*

By applying this theorem we shall prove the following preparatory lemma for the proof of the non-degeneracy of ϱ_H.

Lemma 6.3. *Let (M, α), (N, ω) be as in Claim 1.2. Suppose that there exists $\varphi \in \operatorname{Cont}_0(M, \alpha)$ with $\varrho_\alpha(\operatorname{id}, \varphi) > 0$. In the case (2) we assume, in addition, that $\varphi \in [\operatorname{Cont}_0(M, \alpha), \operatorname{Cont}_0(M, \alpha)]$. Then ϱ_H is a metric.*

Proof. In view of (6.4) $\varrho_H(\operatorname{id}, q(\varphi)) > 0$. Denote $\mathcal{G} = \operatorname{Cont}_0(M, \alpha)$, $\mathcal{H} = \operatorname{Ham}(N, \omega)$. Observe that $\{\psi \in \mathcal{G} : \varrho_\alpha(\operatorname{id}, \psi) = 0\}$ and $\{\psi \in [\mathcal{G}, \mathcal{G}] : \varrho_\alpha(\operatorname{id}, \psi) = 0\}$ are normal subgroups of $\mathcal{G}$ and $[\mathcal{G}, \mathcal{G}]$ resp. It follows that $\{\bar{\psi} \in \mathcal{H} : \varrho_H(\operatorname{id}, \bar{\psi}) = 0\} = \{e\}$ in the case (1), and $\{\bar{\psi} \in [\mathcal{H}, \mathcal{H}] : \varrho_H(\operatorname{id}, \bar{\psi}) =$

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$0\} = \{e\}$ in the case (2), in view of Theorem 6.2. Hence ϱ_H is non-degenerate on $\mathcal{H}$ in the case (1), and ϱ_H is non-degenerate on $[\mathcal{H}, \mathcal{H}]$ in the case (2).

Take any $g \in \mathcal{H}$, $g \neq$ id. There is $p \in N$ with $g(p) \neq p$. Choose a neighborhood U of $p \in N$ such that $g(U) \cap U = \emptyset$, and $h \in \mathcal{H}$ supported in U such that $h(p) \neq p$. Then $gh \neq hg$ and $[g, h] \neq$ id. Therefore $0 < \varrho_H([g, h]) \leq 2\varrho_H(g)$. Thus ϱ_H is non-degenerate on $\mathcal{H}$ itself. $\qquad\square$

7. Proofs of Claim 1.2 and of Theorems 1.3 and 1.4

In this section we shall show simultaneously Claim 1.2, and Theorems 1.3 and 1.4. As a consequence we shall have the non-degeneracy of ϱ_H. The proof will be divided into six parts.

7.1. Definition of a mapping K_α.

Let $\hat{U} = U \times \mathbb{S}^1$ be the prequantization of U, where U is a chart domain and $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Let $f \in \text{Cont}_0(\hat{U}, \alpha_{st})$. Denote by

$$f_t = (g_t, h_t, k_t) = (q(f_t), k_t), \quad \text{where} \quad g_t = (g_t^1, \ldots, g_t^n), \ h_t = (h_t^1, \ldots, h_t^n),$$

a strict contact isotopy supported in $\hat{U}$ such that $f_0 = $ id and $f_1 = f$. Here $k_t : \hat{U} \to \mathbb{S}^1$ is such that $k_t k_0^{-1}$ descends to a map $U \to \mathbb{S}^1$, and $q : \text{Cont}_0(M, \alpha) \to \text{Ham}(N, \omega)$ given by (6.1). Then there exists a unique lift

$$(7.1) \qquad \tilde{f}_t = (g_t, h_t, \tilde{k}_t) = (q(f_t), \tilde{k}_t) : \hat{U} \to U \times \mathbb{R}$$

such that $\tilde{k}_0(p) = 0$ for all p, where $\tilde{k}_t : \hat{U} \to \mathbb{R}$ is the lift of $k_t k_0^{-1} : \hat{U} \to \mathbb{S}^1$. In particular, $\frac{\partial \tilde{k}_t}{\partial t}(p) = \frac{\partial k_t}{\partial t}(p)$ for all p and t. Notice that $\frac{\partial k_t}{\partial t}(p)$ is independent of the choice of a canonical chart on $\hat{U}$ so that it is a well-defined real number. We define for all $p \in \hat{U}$

$$(7.2) \qquad K_\alpha(\{f_t\})(p) := \tilde{k}_1(p) = \int_0^1 \frac{\partial k_t}{\partial t}(p) dt.$$

It follows that K_α is independent of the canonical coordinates in $\hat{U}$. Clearly $K_\alpha(\{f_t\})$ is a smooth map. For any concatenation $\{f_t\} = \{f_t^r\} \star \cdots \star \{f_t^1\}$ with all factors supported in $\hat{U}$ we have the equality for all $p \in \hat{U}$

(7.3)
$$\tilde{k}_t(p) = \tilde{k}_1^1(p) + \tilde{k}_1^2(f_1^1(p)) + \cdots + \tilde{k}_1^{\rho-1}(f_1^{\rho-2} \cdots f_1^1(p)) + \tilde{k}_{rt-\rho+1}^\rho(f_1^{\rho-1} \cdots f_1^1(p)),$$

whenever $t \in [\frac{\rho-1}{r}, \frac{\rho}{r}]$, where $\rho \in \mathbb{N}$, $1 \leq \rho \leq r$. Here $\tilde{k}_t^i$ corresponds to $\{f_t^i\}$ by (7.1). In particular, $K_\alpha(\{f_t\} \star \{f_t'\}) = K_\alpha(\{f_t\}) \circ f_1' + K_\alpha(\{f_t'\})$.

Now we pass to the global case. Let $\pi : (M, \alpha) \to (N, \omega)$ be a prequantization bundle. Given an isotopy $\{f_t\}$ in $\text{Cont}_0(M, \alpha)$ we can define a lift

$$(7.4) \qquad \tilde{f}_t = (q(f_t), \tilde{k}_t) : M \to N \times \mathbb{R}, \quad \text{such that} \quad \tilde{k}_0 = 0,$$

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generalizing the lift (7.1) in the following way. It suffices to define $\tilde{k}_t$. For fixed $p \in M$ take a concatenation $\{f_t\} = \{f_t^{r(p)}\} \star \cdots \star \{f_t^1\}$ with C^1 small factors and with $r(p)$ depending on p such that $\pi(f_t^j(p))$, $t \in I$, lies in a chart domain U_j trivializing π for $j = 1, \ldots, r(p)$. By (7.1) there exists $t \mapsto \tilde{k}_t^j(p) \in \mathbb{R}$ with $\tilde{k}_0^j(p) = 0$. Then we define $\tilde{k}_t$ in (7.4) by making use of the formula (7.3), where now $\{f_t^j\}$ are supported in possibly different $\hat{U}_j$. This definition is independent of the choice of a concatenation for p. In fact, given two such concatenations we can take a common sub-concatenation and use (7.3). It is also independent of the choice of canonical charts in light of a remark before (7.2). In order to generalize (7.2) we define

(7.5) $\qquad K_\alpha : \mathcal{J}_{\mathrm{id}}\, \mathrm{Cont}_0(M, \alpha) \to C^\infty(M, \mathbb{R}), \quad K_\alpha(\{f_t\}) := \tilde{k}_1,$

where $\tilde{k}_t$ is defined as above.

Observe that if $\{f_t\}$ and $\{f_t'\}$ are homotopic rel. endpoints then the resulting $K_\alpha(\{f_t\}), K_\alpha(\{f_t'\}) : M \to \mathbb{R}$ coincide. Indeed, let $\{f_{t,s}\}$ be a homotopy rel. endpoints joining $\{f_t\}$ to $\{f_t'\}$, and let $\tilde{k}^s = K_\alpha(\{f_{t,s}\}) : M \to \mathbb{R}$ corresponds to $\{f_{t,s}\}$ by (7.5) with s fixed. For any $p \in M$ and $s \in I$, by passing to local trivializations of π in neighborhoods of $p, f_{1,s}^1(p), \ldots, (f_{1,s}^{r(s)-1} \cdots f_{1,s}^1)(p)$ we have that the function $s \mapsto \tilde{k}^s(p)$ is locally constant. This is so because we can take concatenations $\{f_{t,s'}\} = \{f_{t,s'}^{r(s)}\} \star \cdots \star \{f_{t,s'}^1\}$ of the isotopies $\{f_{t,s'}\}$, with s' close to s, such that $K_a(\{f_{t,s'}^j\})(f_{1,s'}^{j-1} \cdots f_{1,s'}^1)(p)$, where $j = 1, \ldots, r(s)$, are constant. Therefore, the function $s \mapsto \tilde{k}^s(p)$ is globally constant, as required.

Now, suppose $f_t = f_t^l \circ \cdots \circ f_t^1$ for all t. Since the isotopy $\{f_t\}$ is homotopic rel. endpoints to $\{f_t^l\} \star \cdots \star \{f_t^1\}$ we obtain by (7.5) and (7.3)
(7.6)
$$K_\alpha(\{f_t\})(p) = K_\alpha(\{f_t^1\})(p) + K_\alpha(\{f_t^2\})(f_1^1(p)) + \cdots + K_\alpha(\{f_t^l\})(f_1^{l-1} \cdots f_1^1(p)).$$

Summing-up the above considerations we have

Proposition 7.1. *The mapping* $K_\alpha : \mathcal{J}_{\mathrm{id}}\, \mathrm{Cont}_0(M, \alpha) \to C^\infty(M, \mathbb{R})$ *possesses the following properties:*

(1) $K_\alpha(\{f_t\})$ *is constant on any fiber* $\pi^{-1}(p)$, $p \in N$.
(2) $K_\alpha(\{f_t\} \star \{f_t'\}) = K_\alpha(\{f_t\}) \circ f_1' + K_\alpha(\{f_t'\})$.
(3) *If* $\{f_t\}$ *and* $\{f_t'\}$ *are homotopic rel. endpoints then* $K_\alpha(\{f_t\}) = K_\alpha(\{f_t'\})$.
(4) $K_\alpha(\{f_t\}\{f_t'\}) = K_\alpha(\{f_t\}) \circ f_1' + K_\alpha(\{f_t'\})$. *More generally, if* $f_t = f_t^l \circ \cdots \circ f_t^1$ *for all* t *then (7.6) holds.*
(5) *If for some* $p \in M$ *the map* $t \mapsto f_t(p)$ *is a contractible loop then* $K_\alpha(\{f_t\})(p) = 0$. *In particular, for any open subset* $V \subset N$, *if* $\mathrm{supp}(\{f_t\}) \subset \hat{V}$ *then* $\mathrm{supp}(K_\alpha(\{f_t\})) \subset \hat{V}$.
(6) *If for some* $p \in M$ *the map* $t \mapsto f_t(p)$ *is a loop then* $K_\alpha(\{f_t\})(p) \in \mathbb{Z}$.
(7) $K_\alpha(\{f_t^{-1}\}) = -K_\alpha(\{f_t\}) \circ f_1^{-1}$.

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(8) $K_\alpha(\{f'_t\}\{f_t^{-1}\}) = (K_\alpha(\{f'_t\}) - K_\alpha(\{f_t\})) \circ f_1^{-1}$.

Proof. (1)-(4) are proved above, and (5) follows from (3). (7) is a consequence of (4) and (5), and (8) of (4) and (7). It remains to show (6). In fact, if $t \mapsto f_t(p)$ is a loop, then $k_1(p)k_0(p)^{-1} = 0 \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Hence $\tilde{k}_t$, the lift of $k_t k_0^{-1}$, satisfies $K_\alpha(\{f_t\})(p) = \tilde{k}_1(p) \in \mathbb{Z}$. □

7.2. Construction of a vertical translation φ.
Choose arbitrarily $a > 0$, $b > 0$ and denote $U = (-(a + b), a + b)^{2n}$. Take $c > 0$ and a smooth bump function

$$\mu = \mu_{a,b,c} : [-(a + b), a + b] \to [0, c]$$

such that $\text{supp}(\mu) \subset (-(a+\frac{1}{2}b), a+\frac{1}{2}b)$, $\mu = c$ on $[-a, a]$, $\mu' \geq 0$ on $[-(a+b), 0]$, $\mu' \leq 0$ on $[0, a + b]$, and $|\mu'| < \frac{2c}{b}$. For all $p \in U$, $p = (x, y, z)$, we define

$$(7.7) \qquad F(p) := F_{a,b,c}(p) := \mu(x_1) \cdots \mu(x_n)\mu(y_1) \cdots \mu(y_n).$$

If $\{\phi_t\} := \Psi_\alpha^{-1}(F_{a,b,c})$ then we set $\varphi := \varphi_{a,b,c} := \varphi_1$. Denote $\varphi_t = (q(\varphi_t), k_t^\varphi)$. Assume that $\{f_t\}$ is an arbitrary element of $\mathcal{J}_{\text{id}}^\varphi \text{Cont}_0(\hat{U}, \alpha_{st})$ and that $f_t = (g_t, h_t, k_t)$. Hence, in view of Prop. 7.1, $K_\alpha(\{f_t\}) - K_\alpha(\{\varphi_t\})$ is an integer constant. Since $K_\alpha(\{f_t\})$ and $K_\alpha(\{\varphi_t\})$ are compactly supported in $\hat{U}$, we have $K_\alpha(\{f_t\}) = K_\alpha(\{\varphi_t\})$. It follows that

$$(7.8) \qquad K_\alpha(\{f_t\})(p) = K_\alpha(\{\varphi_t\})(p) \geq 0, \quad \forall p \in \hat{U},$$

$$(7.9) \qquad K_\alpha(\{f_t\})(p) = K_\alpha(\{\varphi_t\})(p) = c^{2n}, \quad \forall p \in \hat{B},$$

where $B := [-a, a]^{2n}$. Indeed, in view of (3.11) we have

$$(7.10) \qquad \frac{\partial k_t^\varphi}{\partial t}(p) = (F - \sum_{i=1}^{n} y_i F_{y_i})(f_t(p)) \geq 0$$

for all p and t, since it is easily seen that $F - \sum_{i=1}^{n} y_i F_{y_i} \geq 0$ on $\hat{U}$. Hence by (7.2) we get (7.8). Next, (7.9) follows from (7.10) and (7.2) since for every $t \in I$ and every $p \in \hat{B}$ we have $\frac{\partial k_t^\varphi}{\partial t}(p) = F(\varphi(p)) = c^{2n}$.

7.3. Estimates in the local case.
Given $\{f_t\} \in \mathcal{J}_{\text{id}} \text{Cont}_0(\hat{U}, \alpha_{st})$, let $F = \Psi_\alpha(\{f_t\})$ be the corresponding Hamiltonian in $C_b^\infty(I \times \hat{U}, \mathbb{R})$. Let $U = (-(a + b), a + b)^{2n}$ as in section 7.2. We have in view of the equality (3.10) for any $p \in \hat{U}$

$$l_{\alpha_{st}}(\{f_t\}) = \int_0^1 \| \ F_t \ \|_\infty \ dt$$

$$\geq \int_0^1 \max_{p \in \hat{U}} |F_t(p)| dt$$

$$\geq \int_0^1 |F_t(f_t(p))| dt$$

(7.11)
$$= \int_0^1 \left| \alpha_{f_t(p)} \left(\frac{\partial f_t}{\partial t}(p) \right) \right| dt$$

$$= \int_0^1 \left| \frac{\partial k_t}{\partial t}(p) - \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p) \right| dt$$

$$\geq \left| \int_0^1 \left(\frac{\partial k_t}{\partial t}(p) - \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p) \right) dt \right|$$

$$= \left| \int_0^1 \frac{\partial k_t}{\partial t}(p) dt - \int_0^1 \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p) dt \right|.$$

It follows from (7.2) and (7.11) that for any $p \in \hat{U}$
(7.12)
$$K_\alpha(\{f_t\})(p) - l_{\alpha_{st}}(\{f_t\}) \leq \int_0^1 \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p) dt \leq K_\alpha(\{f_t\})(p) + l_{\alpha_{st}}(\{f_t\}).$$

From now on U will be identified with $U \times \{0\} \subset U \times \mathbb{S}^1 = \hat{U}$. Then using (7.12), (3.11), (6.8), Prop.7.1(1), bearing in mind that f_t is independent of z,

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and applying Fubini's theorem we have

(7.13)
$$\int_U K_\alpha(\{f_t\})(p)dp \le \int_U \left(\int_I \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p)dt + l_{\alpha_{st}}(\{f_t\}) \right) dp$$

$$= \int_I \left(\int_U \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p)dp \right) dt + \int_U l_{\alpha_{st}}(\{f_t\})dp$$

$$= \int_I \left(\int_{\hat{U}} \sum_{i=1}^n h_t^i(p) \frac{\partial g_t^i}{\partial t}(p)\nu_\alpha \right) dt + \mathrm{vol}(U)l_{\alpha_{st}}(\{f_t\})$$

$$= -\int_I \left(\int_{\hat{U}} \left(\sum_{i=1}^n y_i \frac{\partial F_t}{\partial y_i} \right) (f_t(p))\nu_a \right) dt + \mathrm{vol}(U)l_{\alpha_{st}}(\{f_t\})$$

$$= -\int_I \left(\int_{\hat{U}} \left(\sum_{i=1}^n y_i \frac{\partial F_t}{\partial y_i} \right) (p)\nu_a \right) dt + \mathrm{vol}(U)l_{\alpha_{st}}(\{f_t\})$$

$$\le -\int_I \left(\int_U \left(\sum_{i=1}^n y_i \frac{\partial F_t}{\partial y_i} \right) (p)dp \right) dt + (2a + 2b)^{2n} l_{\alpha_{st}}^\mu(\{f_t\})$$

$$= -\sum_{i=1}^n \int_{U^{(i)}} \left(\int_I \left(\int_{L_{i,\beta}} y_i \frac{\partial F_t}{\partial y_i}(t,p)dy_i \right) dt \right) d(x, y_i', z) + (2a + 2b)^{2n} l_{\alpha_{st}}^\mu(\{f_t\}),$$

where $y_i' = (y_1, \ldots, \hat{y}_i, \ldots, y_n)$, $U^{(i)} = \mathrm{pr}_i(U)$, where pr_i is the canonical projection along y_i, and for any $\beta \in I \times U^{(i)}$ we denote $L_{i,\beta} = \{(t,p) \in I \times U : (t, x, y_i', z) = \beta\}$ which is identified with $(-(a+b), a+b)$. Further, integrating by parts we get

$$\int_0^1 \left(\int_{-(a+b)}^{a+b} y_i \frac{\partial F_t}{\partial y_i} dy_i \right) dt = \int_0^1 \left(y_i F_t |_{-(a+b)}^{a+b} - \int_{-(a+b)}^{a+b} F_t dy_i \right) dt$$

(7.14)
$$\le 0 + (2a + 2b) \int_0^1 \max_{p \in \hat{U}} |F_t(p)|dt$$

$$\le (2a + 2b)l_{\alpha_{st}}^\mu(\{f_t\}).$$

Therefore, in view of (7.13) and (7.14) we have
(7.15)
$$\int_U K_\alpha(\{f_t\})(p)dp \le \mathrm{vol}(U)(n+1)l_{\alpha_{st}}^\mu(\{f_t\}) = (2a + 2b)^{2n}(n+1)l_{\alpha_{st}}^\mu(\{f_t\}).$$

Notice that the first inequality in (7.15) is true as well for an arbitrary canonical chart domain U in view of Fubini's theorem.

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Now suppose $\{f_t\} \in \mathcal{J}_{\mathrm{id}}^\varphi \mathrm{Cont}_0(M, \alpha)$. For $B = [-a, a]^{2n}$ it follows from (7.8), (7.9) and (7.15) that

$$(2a)^{2n} c^{2n} = c^{2n} \,\mathrm{vol}(B) = \int_B c^{2n} dp = \int_B K_\alpha(\{f_t\})(p) dp$$

(7.16)

$$\leq \int_U K_\alpha(\{f_t\})(p) dp \leq (2a + 2b)^{2n} (n+1) l_{\alpha_{st}}^\mu(\{f_t\}).$$

Thus, in light of Lemma 6.1 and (6.4) we obtain the following inequality

(7.17)
$$\frac{a^{2n} c^{2n}}{(a+b)^{2n}(n+1)} \leq \varrho_\alpha^{\hat{U}}(\mathrm{id}, \varphi) = \varrho_H^U(\mathrm{id}, q(\varphi)),$$

where $\varrho_\alpha^{\hat{U}}$ is the metric ϱ_α given by (4.6) for $\hat{U}$, ϱ_H^U is the Hofer metric for U. In general, the above inequality cannot be deduced from the energy-capacity inequality since $\mathrm{supp}(q(\varphi))$ is independent of c.

7.4. **Proof of Theorem 1.3.** Let $\pi : (M, \alpha) \to (N, \omega)$ be a prequantization bundle. Suppose that $N = U_1 \cup \ldots \cup U_r$ is an open covering satisfying the hypotheses of Theorem 1.3, and let $\varphi \in \mathrm{Cont}_{\hat{U}}(M, \alpha)$ be defined by (7.7).

Suppose that $\{f_t\} \in \mathcal{J}_{\mathrm{id}}^\varphi \mathrm{Cont}_0(M, \alpha)$ is an arbitrary isotopy. Then the isotopy $\{\varphi_t f_t^{-1}\}$ is a loop, where $\{\varphi_t\} = \Psi_\alpha^{-1}(F_{a,b,c})$. Therefore, in view of Prop. 7.1,

$$(K_\alpha(\{\varphi_t\}) - K_\alpha(\{f_t\})) \circ f_1^{-1} = K_\alpha(\{\varphi_t f_t^{-1}\}) = m \in \mathbb{Z}.$$

It follows that $K_\alpha(\{f_t\}) = -m$ on $M \setminus \hat{U}$, since f_1 preserves $\hat{U}$. Then, due to (6.5), (6.6) and Lemma 3.3, we can replace $\{f_t\}$ by $\{\bar{f}_t\} := \{\tau_{m,t} f_t\}$, where $\{\tau_{m,t}\} = \Psi_\alpha^{-1}(m)$ and $m \in \mathrm{C}^\infty(M, \mathbb{R})$ is the constant function. It follows that $\{\bar{f}_t\}$ has the same properties and, in addition, $K_\alpha(\{\bar{f}_t\}) = 0$ on $M \setminus \hat{U}$. Clearly $l_\alpha(\{\bar{f}_t\}) = l_\alpha(\{f_t\})$ and $l_\alpha^\mu(\{\bar{f}_t\}) = l_\alpha^\mu(\{f_t\})$. Again by Prop. 7.1 and by the fact that $\mathrm{supp}(\{\bar{f}_t\}) \subset \hat{U}$ it follows that $K_\alpha(\{\bar{f}_t\}) = K_\alpha(\{\varphi_t\})$ on M. Thus we may and do assume that

(7.18)
$$K_\alpha(\{f_t\}) = K_\alpha(\{\varphi_t\}).$$

Observe that (7.18) still holds for N open, even by a simpler argument.

Now, we take a concatenation for $\{f_t\}$

$$\{f_t\} = \{g_t^R\} \star \cdots \star \{g_t^1\}$$

with with sufficiently C^1-small factors $\{g_t^\rho\} \in \mathcal{J}_{\mathrm{id}} \mathrm{Cont}_0(M, \alpha)$. In view of definition (4.2) of the length l_α, we then have

(7.19)
$$l_\alpha(\{f_t\}) = \sum_{\rho=1}^R l_\alpha(\{g_t^\rho\}).$$

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Next, due to Lemma 3.5, there is a fragmentation

$$\forall t \in I, \ \forall \rho = 1, \ldots, r, \quad g_t^\rho = g_t^{\rho,R} \circ \cdots \circ g_t^{\rho,1}, \quad \operatorname{supp}(g_t^{\rho,i}) \subset \hat{U}_i,$$

smoothly depending on t, where $i = 1, \ldots, r$. In view of the usual procedure of fragmentation and Lemma 3.3, we have for all ρ and i

$$(7.20) \qquad\qquad l_\alpha(\{g_t^{\rho,i}\}) \leq 2l_\alpha(\{g_t^\rho\}).$$

Consequently we get a decomposition

$$\forall t \in I, \quad f_t = f_t^l \circ \cdots \circ f_t^1,$$

with each factor $\{f_t^j\}$ supported in $U_{i(j)} \in \mathcal{U}$. Here each $\{f_t^j\}$ identifies with $\{g_t^{\rho,i}\}$ for some $1 \leq \rho \leq R$ and $1 \leq i \leq r$ so that $l = Rr$. In light of (7.19) and (7.20), it follows that

$$(7.21) \qquad \sum_{j=1}^l l_\alpha(\{f_t^j\}) = \sum_{\rho=1}^R \sum_{i=1}^r l_\alpha(\{g_t^{\rho,i}\}) \leq 2r \sum_{\rho=1}^R l_\alpha(\{g_t^\rho\}) = 2r l_\alpha(\{f_t\}).$$

Since $f_1^{j-1} \cdots f_1^1$ are strict contactomorphisms, by using (7.6) we get

$$(7.22)$$

$$\int_M K_\alpha(\{f_t\})(p)\nu_\alpha = \sum_{j=1}^l \int_M K_\alpha(\{f_t^j\})(f_1^{j-1} \cdots f_1^1(p))\nu_\alpha = \sum_{j=1}^l \int_M K_\alpha(\{f_t^j\})(p)\nu_\alpha.$$

Thus, from (7.18), (7.9), (7.8), (7.22), Prop. 7.1, and (7.15) we have

$$(7.23)$$
$$
\begin{aligned}
(2a)^{2n} c^{2n} &= \int_B K_\alpha(\{f_t\})(p)dp \\
&\leq \int_M K_\alpha(\{f_t\})(p)\nu_\alpha \\
&= \sum_{j=1}^l \int_M K_\alpha(\{f_t^j\})(p)\nu_\alpha \\
&= \sum_{j=1}^l \int_{U_{i(j)}} K_\alpha(\{f_t^j\})(p)dp \\
&\leq \sum_{j=1}^l (n+1)\operatorname{vol}(U_{i(j)}) l_{\alpha st}^\mu(\{f_t^j\}) \\
&\leq (n+1)\operatorname{vol}(U) \sum_{j=1}^l l_{\alpha st}^\mu(\{f_t^j\}).
\end{aligned}
$$

For any $\varepsilon > 0$ and any $1 \leq j \leq l$, in view of Lemma 6.1, let $\{\tilde{f}_t^j\} \in \mathcal{J}_{\mathrm{id}} \operatorname{Cont}_0(M, \alpha)$ with $\operatorname{supp}(\{\tilde{f}_t^j\}) \subset U_{i(j)}$ such that $\tilde{f}_1^j = f_1^j$ and it satisfies the inequality

$$(7.24) \qquad l_{\alpha_{st}}^\mu(\{\tilde{f}_t^j\}) \leq l_{\alpha_{st}}(\{f_t^j\}) + \frac{\varepsilon}{l}.$$

Replace $\{f_t\}$ by $\{\tilde{f}_t\}$ such that $\tilde{f}_t := \tilde{f}_t^l \circ \cdots \circ \tilde{f}_t^1$ for all t. Then $\tilde{f}_0 = \mathrm{id}$ and $\tilde{f}_1 = \varphi$. From (7.23), (7.24) and (7.21) we obtain

$$(2a)^{2n} c^{2n} \leq (n+1) \operatorname{vol}(U) \sum_{j=1}^l l_{\alpha_{st}}^\mu(\{\tilde{f}_t^j\})$$

$$\leq (n+1) \operatorname{vol}(U) \sum_{j=1}^l \left(l_{\alpha_{st}}(\{f_t^j\}) + \frac{\varepsilon}{l} \right)$$

$$\leq (2n+2) r (2a + 2b)^{2n} l_{\alpha_{st}}(\{f_t\}) + \varepsilon.$$

Therefore

$$(7.25) \qquad (2a)^{2n} c^{2n} \leq (2n+2) r (2a + 2b)^{2n} l_{\alpha_{st}}(\{f_t\}).$$

Consequently, in view of (6.4), we get for $\varphi = \varphi_{a,b,c} \in \operatorname{Cont}_0(M, \alpha)$ and for $q(\varphi) \in \operatorname{Ham}(N, \omega)$

$$(7.26) \qquad \frac{a^{2n} c^{2n}}{(a+b)^{2n}(2n+2) r} \leq \varrho_\alpha(\mathrm{id}, \varphi) = \varrho_H(\mathrm{id}, q(\varphi)).$$

7.5. **Proof of Theorem 1.4.** . The proof of Theorem 1.3 is still valid for Theorem 1.4 with some modifications.

By standard arguments we can find an open cover $N = U_0 \cup \ldots \cup U_{2n}$ with $U_i = \bigsqcup_k U_{i,k}$, where $i = 0, \ldots, 2n$, $k = 1, 2, \ldots$, such that the cover $\mathcal{U} = \{U_{i,k}\}$ is locally finite and the family $\{U_{i,k}\}_{k=1}^\infty$ is pairwise disjoint for all i. We can arrange so that $U = U_{0,1}$ is such that $\operatorname{vol}(U) = \max_{i,k} \operatorname{vol}(U_{i,k})$. Moreover, we assume that $\mathcal{U}$ is a good cover, that is each intersection of elements of $\mathcal{U}$ is a ball.

Let $\varphi = \varphi_{a,b,c} \in \operatorname{Cont}_{\hat{U}}(M, \alpha)$ be defined by $F_{a,b,c}$ given by (7.7). Choose arbitrarily $\{f_t\} \in \mathcal{J}_{\mathrm{id}}^\varphi \operatorname{Cont}_0(M, \alpha)$. For the case of (N, ω) open there is a compact subset $C \subset N$ such that the isotopy $q(\{f_t\})$ is supported in C and we use Lemma 3.5 with this C. Consequently, in each case (1) or (2) we may have that

$$\operatorname{supp}(\{f_t\}) \subset \bigcup_{i=0}^{2n} \bigsqcup_{k=1}^{k(i)} U_{i,k}$$

for some $k(i) \in \mathbb{N}$, $i = 0, \ldots, 2n$.

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As before we take a concatenation $\{f_t\} = \{g_t^r\} \star \cdots \star \{g_t^1\}$ with with sufficiently C^1-small factors $\{g_t^\rho\} \in \mathcal{J}_{\mathrm{id}} \mathrm{Cont}_0(M, \alpha)$ such that (7.19) holds true. By Lemma 3.5, for all t and ρ there is a fragmentation

$$(7.27) \qquad g_t^\rho = g_t^{\rho,2n} \circ \cdots \circ g_t^{\rho,0}, \quad \mathrm{supp}(g_t^{\rho,i}) \subset \hat{U}_i, \quad i = 0, \ldots, 2n,$$

smoothly depending on t, with (7.20). Next,

$$(7.28) \qquad \forall \rho = 1, \ldots, R, \ \forall i = 0, \ldots, 2n, \quad g_t^{\rho,i} = g_t^{\rho,i,k(i)} \circ \cdots \circ g_t^{\rho,i,1},$$

where $\mathrm{supp}(g_t^{\rho,i,k}) \subset U_{i,k}$ for $k = 1, \ldots, k(i)$. Thus we have a decomposition

$$(7.29) \qquad \forall t \in I, \quad f_t = f_t^l \circ \cdots \circ f_t^1,$$

with each factor $\{f_t^j\}$ supported in $U_{i(j),k(j)}$. In (7.29) each $\{f_t^j\}$ is identified with some $\{g_t^{\rho,i,k}\}$. It follows that $l = R \sum_{i=0}^{2n} k(i)$. Observe that the decomposition (7.29) consists of blocks of the type (7.27) and, due to (7.28), actually of blocks of the type (7.28). We shall analyze the formula (7.22) with respect to these blocks.

Denote $\tilde{g}^{\rho,i} = g_1^{\rho,i-1} \cdots g_1^{\rho,0} g_1^{\rho-1} \cdots g_1^1$ and $U^{\rho,i} = \tilde{g}_1^{\rho,i}(U)$. Then, in view of Prop. 7.1 and (7.6) we have

$$
\begin{aligned}
\int_M K_\alpha(\{f_t\})(p)\nu_\alpha &= \int_U K_\alpha(\{f_t\})(p)dp \\
&= \sum_{j=1}^{l} \int_U K_\alpha(\{f_t^j\})(f_1^{j-1} \cdots f_1^1(p))dp \\
(7.30) \qquad\qquad &= \sum_{\rho,i} \sum_{k=1}^{k(i)} \int_U K_\alpha(\{g_t^{\rho,i,k}\})(g_t^{\rho,i,k-1} \cdots g_t^{\rho,i,1}\tilde{g}^{\rho,i})(p)dp \\
&= \sum_{\rho,i} \sum_{k=1}^{k(i)} \int_{U^{\rho,i}} K_\alpha(\{g_t^{\rho,i,k}\})(g_t^{\rho,i,k-1} \cdots g_t^{\rho,i,1})(p)dp
\end{aligned}
$$

Now observe that $\mathrm{vol}(U^{\rho,i}) = \mathrm{vol}(U)$ and that for all t, ρ, i the supports of $g_t^{\rho,i,k(i)}, \ldots, g_t^{\rho,i,1}$ are pairwise disjoint. It follows that

$$(7.31) \qquad \int_{U^{\rho,i}} K_\alpha(\{g_t^{\rho,i,k}\})(g_t^{\rho,i,k-1} \cdots g_t^{\rho,i,1})(p)dp = \int_{U^{\rho,i,k}} K_\alpha(\{g_t^{\rho,i,k}\})(p)dp,$$

where $U^{\rho,i,k}$, $k = 1, \ldots, k(i)$, are pairwise disjoint balls such that $\sum_{i=1}^{k(i)} \text{vol}(U^{\rho,i,k}) \leq$ $\text{vol}(U)$. Therefore, due to (7.30), (7.31) and (7.15)

$$
\int_M K_\alpha(\{f_t\})(p)\nu_\alpha = \sum_{\rho,i} \sum_{k=1}^{k(i)} \int_{U^{\rho,i,k}} K_\alpha(\{g_t^{\rho,i,k}\})(p)dp
$$

$$
\leq \sum_{\rho,i} \sum_{k=1}^{k(i)} (n+1)\, \text{vol}(U^{\rho,i,k}) l_{\alpha_{st}}^\mu(\{g_t^{\rho,i,k}\})
$$

(7.32)

$$
\leq (n+1) \sum_{\rho,i} \sum_{k=1}^{k(i)} \text{vol}(U^{\rho,i,k}) l_{\alpha_{st}}^\mu(\{g_t^{\rho,i}\})
$$

$$
\leq (n+1)\, \text{vol}(U) \sum_{\rho,i} l_{\alpha_{st}}^\mu(\{g_t^{\rho,i}\}).
$$

Finally, using (7.32) and proceeding exactly as at the end of section 7.4 we obtain

$$
(2a)^{2n}c^{2n} \leq (4n^2 + 6n + 2)(2a + 2b)^{2n} l_{\alpha_{st}}(\{f_t\})
$$

and due to (6.4)

(7.33) $$\frac{a^{2n}c^{2n}}{(a+b)^{2n}(4n^2 + 6n + 2)} \leq \varrho_\alpha(\text{id}, \varphi) = \varrho_H(\text{id}, q(\varphi)).$$

7.6. **Construction of ψ with good properties.** By good properties we mean (7.37) and (7.41) below and the fact that ψ belongs to the commutator subgroup. For $a > 0$ we denote subsets of $\mathbb{R}^{2n}$ as follows:

$$U_a = [-a, a]^{2n}, \quad V_a = [-3a, 5a] \times [-3a, 3a]^{2n-1}, \quad B_a = [2a, 5a] \times [-3a, 3a]^{2n-1}.$$

Set $U = U_{15a}$. Let $\tau_a \in \text{Cont}_0(\hat{U}, \alpha_{st})$ such that $\tau_a|_{V_a} = \tau_{9a}^{(x_1)}|_{V_a}$ (c.f. section 3), and $\text{supp}(\tau_a) \subset (-4a, 15a) \times (-4a, 4a)^{2n-1}$. That is, on V_a the map τ_a the $9a$-translation along the x_1-axis. Next, for $c > 0$, $d > 0$ by using notation (7.7) we set

$$\chi_{a,c} := [\varphi_{5a,a,c}, \tau_a] = \tau_a^{-1}\varphi_{5a,a,c}^{-1}\tau_a\varphi_{5a,a,c}, \quad \psi_{a,c,d} := [\varphi_{a,a,d}\varphi_{5a,a,c}, \tau_a].$$

It is easily seen that

(7.34) $$\psi_{a,c,d}|_{U_a} = \varphi_{a,a,c+d}|_{U_a},$$

(7.35) $$\psi_{a,c,d}|_{B_a} = \varphi_{5a,a,c}|_{B_a},$$

(7.36) $$\psi_{a,c,d}|_{\backslash U_{5a}} = \chi_{a,c}|_{\backslash U_{5a}}.$$

Define

$$\psi := \psi_{a,c,d}, \quad \psi_t := [\bar{\varphi}_t\varphi_t, \tau_a],$$

where $\{\varphi_t\} := \Psi_\alpha^{-1}(F_{5a,a,c})$ and $\{\bar{\varphi}_t\} := \Psi_\alpha^{-1}(F_{a,a,d})$. Let us take any isotopy $\{f_t\} \in \mathcal{J}_{\text{id}}^\psi \text{Cont}_0(M, \alpha_{st})$. Then, similarly as (7.18), we get $K_\alpha(\{f_t\}) =$

$K_\alpha(\{\psi_t\})$. In view of (7.8), (7.9), (7.34), (7.35), (7.36) and Lemmata 3.3 and 7.1 it follows that

$$(7.37) \qquad K_\alpha(\{f_t\})(p) = K_\alpha(\{\psi_t\})(p) = c^{2n}, \qquad\qquad \forall p \in \hat{B}_a,$$

$$(7.38) \qquad K_\alpha(\{f_t\})(p) = K_\alpha(\{\psi_t\})(p) = (c + d)^{2n}, \qquad \forall p \in \hat{U}_a,$$

$$(7.39) \qquad K_\alpha(\{f_t\})(p) = K_\alpha(\{\psi_t\})(p) \geq 0, \qquad\qquad \forall p \in \hat{U}_{5a},$$

and

$$(7.40) \qquad\qquad K_\alpha(\{f_t\})|_{\backslash \hat{U}_{5a}} \quad \text{is independent of } d.$$

Then (7.38), (7.39) and (7.40) imply that

$$(7.41) \qquad \int_{U \backslash B_a} K_\alpha(\{f_t\})(p)dp = \int_{U \backslash B_a} K_\alpha(\{\psi_t\})(p)dp \geq 0,$$

provided d is large enough.

In view of (7.37) and (7.41) the inequality analogous to (7.26) still holds in our case if we replace φ by ψ. In fact, the inequalities (7.16) and (7.25) remain true in our case. Likewise for (7.33). Since $\mathrm{vol}(B_a) = \frac{1}{3}(6a)^{2n}$ and $\mathrm{vol}(U) = (30a)^{2n}$, we have that $\varrho_\alpha(\mathrm{id}, \psi)$ is bounded from below by $\frac{c^{2n}}{5^{2n}(6n+6)r}$ or by $\frac{c^{2n}}{5^{2n}(12n^2+18n+6)}$. This completes the proof of Claim 1.2.

Proof of the non-degeneracy of ϱ_H. It follows from Claim 1.2 combined with Lemma 6.3.

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